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PREDUALS OF BANACH LATTICES, WEAK ORDER UNITS AND THE RADON-NIKODYM PROPERTY

BY

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INTRODUCTION

In [12] R. C. James proved the following assertions for E a Banach space with an unconditional basis $\{e_n\}_n$:

(WSC) *E is weakly sequentially complete if and only if E contains no isomorphic copy of* $\mathbf{c}_0$;

(R) *E is reflexive if and only if E contains no isomorphic copy of* $\mathbf{c}_0$ *and* $\mathbf{l}_1$;

(RNP) *E' is separable if and only if E contains no isomorphic copy of* $\mathbf{l}_1$.

Later Lozanovski [19], [20] proved (WSC) and (R) for Banach lattices but his approach covers also the case where E embeds into a σ-complete Banach lattice having order continuous norm. Related results were discussed by Lotz [17], Meyer-Nieberg [21] and Tzafriri [29]. In a conversation, in 1974, H. Lotz showed us a proof that (RNP) holds for every separable Banach lattice.

The first section of our paper is concerned with preduals of Banach lattices. A Banach space F is said to be a predual of the Banach space E if F' is isometric to E. The main result (see Theorem 1.1 below, asserts that given an ordered Banach space E which contains no isomorphic copy of $\mathbf{c}_0$ then E' is a Banach lattice if and only if E itself is a Banach lattice. Particularly (WSC) and (R) both remain valid in the framework of ordered preduals of Banach lattices. It is proved also that if E does not contain an isomorphic copy of $\mathbf{c}_0$ and E' is a Banach lattice then E is the unique (up to isometry) ordered predual of E'. That extends the well known fact that a space $L_\infty(\mu)$ has a unique (up to isometry) predual.

In the second section we discuss a geometrical condition in order that the dual of a Banach lattice E having a weak order unit (i.e a total element) fails such a unit: the presence in E' of a lattice isomorph of $\mathbf{l}_1(\Gamma)$ for Γ an uncountable set of indices. See Corollary 2.5 below. Under additional assumptions this condition is seen to be equivalent to the fact that E' is not weakly compactly generated. See Theorem 2.7 below. That extends an important result due to Rosenthal [25].

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In the third section we establish (RNP) for separable Banach spaces E having local unconditional structure in the sense of Gordon and Lewis [6] i.e. E'' is complemented in a Banach lattice. See Theorem 3.3 below. As a consequence we obtain that the recent example (due to R. C. James [13]) of a somewhat reflexive Banach space with a nonseparable dual fails local unconditional structure.

The main results presented in the second and the third section have been announced in [22].

1. PREDUALS OF BANACH LATTICE

We shall denote by $\mathscr{L}_0$ the class of all Banach spaces E equipped with a closed cone C such that;

L_1) *(E, C) has the Riesz decomposition property*

L_2) *$-y \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$*

L_3) *For each $x \in E$, $\|x\| < 1$ there exists a $y \in E$ with $\|y\| < 1$, $y \geqslant \pm x$.*

Here $x \geqslant 0$ means precisely that $x \in C$.

By Theorem 1 in [14], page 18, E' is order isometric to a Banach lattice. It is also well known (see [3]) that each ordered Banach space E whose topological dual is a Banach lattice satisfies $L_1 - L_3$ for

$$C = \{x \in E\,;\ x'(x) \geqslant 0 \text{ for all } x' \in E', x' > 0\}$$

The class $\mathscr{L}_0$ was investigated especially in connection with the study of $L_1(\mu)$-preduals spaces. See [16] for details. Our approach is based on the general theory of AM and AL spaces in the sense of Kakutani.

Let $E \in \mathscr{L}_0$. For each $x \in E$, $x > 0$, we can consider the following vector space:

$$E_x = \{y \in E\,;\, (\exists)\ \lambda > 0,\ \lambda x \geqslant \pm y\}$$

normed by:

$$\|y\|_x = \inf\ \{\lambda > 0\,;\ \lambda x \geqslant \pm y\}$$

By Theorem 6 in [14], page 16, $(E_x)'$ is order isometric to a space $L_1(\mu)$, for μ a suitable positive Radon measure.

If E is supposed to be a Banach lattice then a classical result due to Kakutani yields that E_x is lattice isometric to a $C(S)$ space.

We shall denote by $i_x : E_x \to E$ the canonical inclusion.

For each $x' \in E'$, $x' > 0$, consider on E the following relation of equivalence:

$$x \sim 0 \Leftrightarrow (\forall)\ \varepsilon > 0\ (\exists)\ y_\varepsilon \geqslant \pm x,\ x'(y_\varepsilon) \leqslant \varepsilon.$$

The completion of $E/\sim$ with respect to the additive norm:

$$\|x\|_{x'} = \inf\{x'(y)\,;\, y \geqslant \pm x\}$$

will be denoted by $L_1(x')$ and the canonical mapping $E \to L_1(x')$ by $j_{x'}$. Notice that $(j_{x'})' = i_{x'}$, and $L_1(x')'$ is order isometric to $(E')_{x'}$ for each $x' \in E', x' > 0$.

In [8] (see also [14]) page 96) Grothendieck remarked that each dual $C(S)$ space has a unique (up to isometry) predual. This fact together a classical result due to Kakutani shows that each $L_1(x')$ space is order isometric to an $L_1(\mu)$ space, for μ a suitable positive Radon measure. Another useful remark is that for each $x \in E$, $x > 0$, there exists an order isomorphism ρ_x from $L_1(x)$ into $(E_x)'$ satisfying the following two conditions :

$$\rho_x \circ j_x = (i_x)'.$$

$$\| \rho_x \| = \| (\rho_x)^{-1} \| = 1$$

Our next result shows that the non lattice $\mathscr{L}$—theory requires the presence of $\mathbf{c}_0$:

1.1. THEOREM. *Let $E \in \mathscr{L}_0$ be a Banach space which contains no isomorphic copy of $\mathbf{c}_0$. Then :*

(i) *E is order isometric to a Banach lattice*

(ii) *E' has an unique (up to isometry) ordered predual.*

Proof. (i) Let $x \in E$, $x > 0$. Because E contains no isomorphic copy of $\mathbf{c}_0$, a result due to Lindenstrauss and Tzafriri (see [16], page 184) implies that the mapping i_x is weakly compact and thus $(i_x)'' (E_x)'' \subset E$. On the other hand $(i_x)'' = (j_x)' \circ (\rho_x)'$ and $(\rho_x)'$ is onto. Then :

$$\mathrm{Im}\,(j_x)' \subset \overline{\mathrm{Im}\,(i_x)''} \subset E$$

the closure being considered in the norm topology of E. In other words the ideal $(E'')_x$, generated by x in E'', is contained in E and thus the modulus (calculated in E'') of each $x \in E$ belongs also to E.

(ii). Let Z be the closed subspace of all order continuous functionals $x'' \in E''$ i.e.

$$x'_a \downarrow 0 \text{ (in order) implies } x''(x'_a) \to 0$$

and let Y be an ordered predual of E'. Then there exists an isometry $\varphi : Y \to Z$ given by :

$$\varphi(y)(x') = \langle y, x' \rangle$$

for every $y \in Y$, $x' \in E'$. By (i) and *Proposition 2.4 (d)* in [17] it follows that $E = Z$, which in turn implies that Y is isometric to E, q.e.d.

1.2. COROLLARY. *Let E be an ordered Banach space which contains no isomorphic copy of $\mathbf{c}_0$. Then E' is isometric to a Banach lattice if and only if E itself is isometric to a Banach lattice.*

After this paper has been accepted for publication, Professor Lacey has informed us of the following simple proof of Corollary 1.2: Let $x \in E$ and look at all upper bounds of $\{x, 0\}$; this collection is downwards directed by the interpolatory property. If it does not converge then there is an $\varepsilon > 0$ and a decreasing sequence $\{x_n\}_n$ so that $\|x_{n+1} - x_n\| \geqslant \varepsilon$ for all n. It follows that $\mathrm{Sp}\,\{x_{n+1} - x_n\}_n \sim \mathbf{c}_0$, q.e.d.

The Banach lattice $Jh = (\sum \oplus \mathbf{l}_\infty(n))_{\mathbf{l}_1}$ is weakly sequentially complete and thus, by our Theorem 1.1 (ii) above, Jh is the only ordered predual of $(Jh)' = (\sum \oplus \mathbf{l}_1(n))_{\mathbf{l}_\infty}$. However $(Jh)'$ contains a complemented copy of $\mathbf{l}_1$ (see W. B. Johnson, Israel J. Math. **13** (1972), 301–310), and, $\mathbf{l}_1$ fails an unique predual. In connection with this example we ask the following:

1.3. PROBLEM. *Does there exist a Banach lattice E such that E contains an isomorph of $\mathbf{c}_0$ and E' has a unique (up to isometry) predual?*

2. WEAK ORDER UNITS

In this section we discuss a geometrical condition for the existence of a weak order unit in the dual of a separable Banach lattice: the non existence of a lattice isomorph of a space $\mathbf{l}_1(\Gamma)$ for Γ an uncountable set.

We need a preliminary result which works for all separable spaces in $\mathscr{L}$ and also for all Banach lattice having a weak order unit. Here $\mathscr{L}$ denotes the class of all preduals of Banach lattices.

2.1. LEMMA. *Let $E \in \mathscr{L}$ such that for a suitable $v \in E''$ we have:*

$$x \in E, |x| \wedge |v| = 0 \text{ implies } x = 0.$$

Then there exists an order complete Banach lattice $M(E)$ with a weak order unit and a lattice isometry $i : E' \to M(E)'$ such that:

(a) *$i(E')$ is complemented in $M(E)'$,*

(b) *$i(E')$ is formed by order continuous functionals on $M(E)$.*

Thus $M(E)$ plays the same role as $L_\infty[0.1]$ for $E = C[0,1]$ and (b) extends a well known lemma due to Dini.

Proof. We shall consider for $M(E)$ the band generated by v in E''. Then a lattice isometry $i : E' \to M(E)'$ is given by:

$$i(x')(e) = \langle x', e\rangle$$

for every $x' \in E'$, $e \in M(E)$. Let $j : E \to M(E)$ the canonical inclusion. Then

$$(j' \circ i)(x')(x) = \langle i(x'), j(x)\rangle = \langle x, x'\rangle$$

for every $x \in E$, $x' \in E'$, which implies the existence of a positive projection $P : M(E)' \to i(E')$.

The second assertion is an easy consequence of the following result due to Riesz: if Z is a Banach lattice and $f_n \downarrow 0$ in Z' then $f_n(z) \to 0$ for every $z \in Z$, q.e.d.

If Z is an order complete Banach lattice and $A \subset Z$ is a closed subspace, we shall denote by $\Sigma(A)$ an order complete closed sublattice of the band generated by A such that $A \subset \Sigma(A)$.

2.2. THEOREM. *Let $E \in \mathscr{L}$ be a Banach space which is contained in the band generated by a positive $v \in E''$. If A is a closed subspace of E' then either* :

(i) $\Sigma(A)$ *has a weak order unit*; *or*

(ii) *A contains an isomorph of $\mathbf{l}_1(\Gamma)$ (for Γ an uncountable set) which is complemented in $\Sigma(A)$ and $\Sigma(A)$ contains a lattice isomorph of $\mathbf{l}_1(\Gamma)$.*

Proof. By *Lemma* 2.1 above we can assume that E is an order complete Banach lattice with a weak order unit u and A is formed by order continuous functionals. The subspace E'_0 of all order σ-continuous functionals $x' \in E'$ constitutes a band (see [28] page 74) and thus $\Sigma(A) \subset E'_0$.

By Zorn' s lemma there exists a family $\{u'_i\}_{i\in I}$ of pairwise disjoint normalized elements of $\Sigma(A)$ such that :

$$x' \in \Sigma(A), \sup(|x'| \wedge |u'_i|) = 0 \textit{ implies } x' = 0.$$

Put :

$$H = \{i \in I\,;\,(\exists)\, a_i \in A, \|a_i\| \leqslant 1, [u'_i]\, a_i \neq 0\}$$

and for each $i \in H$, choose an $a_i \in A$ with $\|a_i\| \leqslant 1$ and $[u'_i]a_i \neq 0$. Here $[z]$ denotes the band projection generated by z.

Notice that $x = \sup(x \wedge nu)$ for each $x \in E$, $x > 0$, and $[u'_i]a_i \in E'_0$ for each $i \in H$. Then for each $i \in H$ we can find an $e_i \in [0, u]$ with $([u'_i]a_i)_{e_i} \neq 0$.

The following two possibilities occur :

(1) *Card* $H \leqslant \aleph_0$ and thus by identifying H as a subset of $\mathbb{N}$ we shall denote :

$$u' = \sum_{n\in H} 2^{-n}|u'_n|.$$

Then $u' \in \Sigma(A)$ and :

$$x' \in A, |x'| \wedge u' = 0 \text{ implies } x' = 0.$$

Because $\Sigma(A) \subset A^{\perp\perp}$, it follows that u' is a weak order unit for $\Sigma(A)$.

(2) *Card* $H > \aleph_0$ and in this case we shall consider the operator $T \in \mathbf{L}(\Sigma(A), \mathbf{l}_1(H))$ given by :

$$T(x') = \{([u'_i]\, x')e_i\}_{i\in H}$$

for every $x' \in \Sigma(A)$. Notice that $\|T\| \leqslant \|u\|$ and $T(a_i) \neq 0$, $i \in H$. Put :

$$H_n = \{i \in H\,;\, |T(a_i)(i)| \leqslant 1/n\}$$

for $n = 1,2, \ldots$ Then $H = \cup H_n$ and there exists an n_0 with *Card* $H_{n_0} > \aleph_0$. By *Lemma* 1.1 in [25], A contains a non separable $\mathbf{l}_1(\Gamma)$ space which is complemented in $\Sigma(A)$.

Notice that $T([u_i']a_i) \neq 0$ for each $i \in H$ and the elements $[u_i']a_i$ are pairwise disjoint. Then the proof of Lemma 1.1 in [25] easily yields the existence of an uncountable family

$$\{z_\omega\}_{\omega\in\Omega} \subset Abco\,\{[u_i']a_i\,;\, i \in H\}$$

which is equivalent to the vector unit basis of $\mathbf{l}_1(\Omega)$. Here $Abco\, Z$ means the absolute convex hull of Z. Put:

$$J = \{i \in H\,;\ (\exists)\ \omega \in \Omega,\ |z_\omega| \wedge |[u_i']a_i| \neq 0.$$

Then $Card\, J > \aleph_0$ (otherwise $\{z_\omega\}_\omega$ would be contained in a separable space) and because each z_ω is a finite combination of the elements $[u_i']a_i$, there exists an uncountable subset of $\{z_\omega\}_\omega$ formed by pairwise disjoint elements, q.e.d.

2.3. Remark. *The two possibilities of Theorem 2.2 are mutually exclusive.* In fact, by *Lemma* 2.1 it suffices to consider the case where E is an order complete Banach lattice with a weak order unit $u>0$ and u' is a positive element of E_0' such that $[u']$ contains a pairwise disjoint normalized family $\{e_\gamma'\}_{\gamma\in\Gamma}$ which is equivalent to the unit vector basis of a non separable $\mathbf{l}_1(\Gamma)$. Since $u' \in E_0'$, $([u'| \wedge |e_\gamma'|)u \neq 0$ for each $\gamma \in \Gamma$. Then there exists an $n_0 \in \mathbb{N}$ and an uncountable subset $\Gamma_0 \subset \Gamma$ with $(u' \wedge |e_\gamma'|)u \geqslant 1/n_0$ for each $\gamma \in \Gamma$. Consequently:

$$u'(u) \geqslant \sup\{\sum_{\gamma\in F}(u' \wedge |e_\gamma'|)u\,;\, Card\, F < \infty\} = \infty$$

contradiction.

2.4. Remark. *It is possible that E and E′ both have a weak order unit and E′ contains a complemented isomorph (but not a lattice isomorph) of* $\mathbf{l}_1(2^{\mathbb{N}})$. For example, consider $E = (\Sigma \oplus \mathbf{l}_\infty(n))_{\mathbf{l}_1}$. See [11] for details.

2.5. Corollary. *Let E be a Banach lattice with a weak order unit. Then either E′ contains a weak order unit or E′ contains a lattice isomorph of a non separable* $\mathbf{l}_1(\Gamma)$ *space.*

Professor R. G. Bartle has kindly informed us that H. P. Lotz and H. Rosenthal (Urbana) have obtained results related to Corollary 2.5 above.

2.6. Corollary. *Let E be an order complete Banach lattice such that E′ has a weak order unit. Then either E contains a weak order unit or E contains a lattice isomorph of a non separable* $\mathbf{l}_1(\Gamma)$ *space.*

A special case of our Theorem 2.2 is the following:

2.7. Theorem. *Let* $E \in \mathscr{L}$ *be a Banach space which is contained in the band generated by a suitable* $v \in E''$. *If E contains no complemented copy of* $\mathbf{l}_1$ *and A is a closed subspace of E′ then either:*

(i) *A is contained in a weakly compactly generated sublattice of E′ having a weak order unit; or,*

(ii) *A contains an isomorph of a non separable* $\mathbf{l}_1(\Gamma)$ *space that is complemented in E′.*

Proof. By [1], E' contains no isomorphic copy of $\mathbf{c}_0$ and thus by Propositions 2.1 and 2.4 in [17] the order intervals of E' are relatively weakly compact and the topology of E' is order continuous i.e. $x'_n \downarrow 0$ (in order) implies $\|x'_n\| \to 0$.

Then $[z] = z^{\perp\perp} = \overline{\text{Span}}\ [0, z]$ for each $z \in E'$, $z > 0$, and our result follows from Theorem 2.2 above, q.e.d.

2.8. Remark. If E is a Banach lattice and u' is a positive element of E' then $[u']$ *contains precisely those functionals* $x' \in E'$ *which are absolutely continuous with respect to* u', *i.e.*

(AC) *For every* $\varepsilon > 0$ *and every* $x \in E$, $x > 0$, *there exists a* $\delta = \delta(\varepsilon, x) > 0$ such that :

$$|y| < x, \quad u'(|y|) < \delta \ \textit{ implies } |x'(y)| < \varepsilon.$$

See [2] for details. Consequently for E a $C(S)$ space our Theorem 2.7 above implies Lemma 1.3 in [25].

3. THE RADON-NIKODYM PROPERTY

In 1975, at the Kent University conference on the Radon-Nikodym property, H. Lotz proved the following result :

3.1. Theorem. *Let E be a Banach lattice such that E' has a weak order unit and let A be a closed subspace of E. Then either* :

(i) *A contains an isomorphic copy of* $\mathbf{c}_0$; *or*

(ii) *A' is weakly compactly generated.*

We next present similar results in the setting of Banach spaces having local unconditional structure.

The *unconditional basis constant* $\chi(E)$ of a Banach space E is the least constant λ having the following property : there exists a basis $\{e_i\}_{i\in I}$ for E such that $\|\Sigma \alpha_i a_i e_i\| \leqslant \lambda$ whenever $\Sigma\, a_i e_i \in E$ has norm one and $|\alpha_i| \leqslant 1$, $i \in I$. If not, such λ exists, $\chi(E) = \infty$.

3.2. Definition. *A Banach space E is said to have local unconditional structure (l.u.st.) in the sense of Gordon and Lewis* [6] *if E satisfies one of the following equivalent conditions* :

(i) *There exists a $\lambda > 0$ such that for any finite dimensional subspace $F \subset E$ one can find a space U and operators $\alpha \in \mathscr{L}(F, U)$, $\beta \in \mathscr{L}(U, E)$ such that $\beta \circ \alpha$ is the identity on F and*

$$\|\alpha\|\ \|\beta\|\ \chi(U) \leqslant \lambda,$$

(ii) *E'' is complemented in a Banach lattice*

(iii) *There exists an isomorphism h from E into a Banach lattice L and a $\varphi \in \mathscr{L}(E', L')$ with $h' \circ \varphi = 1_E$.*

The equivalence (i) $\Leftrightarrow$ (ii) was remarked in [5] while (ii) $\Leftrightarrow$ (iii) is immediate.

A stronger concept of *l.u.st.* was introduced by Dubinski, Pelczynski and Rosenthal. See [5] for details.

3.3. THEOREM. *For a separable Banach space E having l.u.st. the following assertions are equivalent* :

(i) *E' is separable*

(ii) *E does not contain a copy of $\mathbf{l}_1$*

(iii) *E' is weakly compactly generated.*

Proof. (i) $\Rightarrow$ (ii). In fact, if a Banach space Z contains an isomorph of $\mathbf{l}_1$ then Z' has a quotient space isomorphic to l_∞ which in turn implies that Z' is nonseparable.

(ii) $\Rightarrow$ (iii). Let h, φ and L be as in Definition 3.2 (iii) above. Since E is separable, we may assume that L is separable.

Since E is separable and contains no isomorphic copy of $\mathbf{l}_1$, E' contains no isomorphic copy or $\mathbf{l}_1(\Gamma)$ for Γ an uncountable set (see [23]),so by our Theorem 2.2 above it follows that $\varphi(E')$ is contained in the band generated by a positive $u' \in L'$.

A result due to Bessaga and Pelczynski [1] implies that E' contains no isomorphic copy of $\mathbf{c}_0$ and thus the composition $h_{x'} = h' \circ i_{x'} : (L')_{x'} \to E'$ is weakly compact for each $x' \in L'$, $x' > 0$. See [26], Theorem 3.7.

Since $(L')_{x'} = (L_1(x'))'$ has the Dunford-Pettis property, $h_{x'}$ maps decreasing sequences of positive elements of $(L')_{x'}$ into converging sequences of elements of E'. See Proposition 1 and Theorem 1 in [7]. On the other hand $h_{x'} = (j_{x'} \circ h)'$ is w'-continuous and thus $h_{x'}$ is order σ-continuous for each $x' > 0$. Consequently :

$$E' = \overline{Span}\, h'[0, u'] = \overline{Span}\, (h_{u'})\ [0,u']$$

and we already remarked that $h_{u'}$ is weakly compact.

The implication (iii) $\Rightarrow$ (i) is an easy consequence of the following result due to Davis, Figiel, Johnson and Pelczynski [4] : a Banach space Z is weakly compactly generated if there exists a reflexive space R and a one to one operator $T \in \mathscr{L}(R, Z)$ with $T(R)$ dense in Z, q. e.d.

3.4. COROLLARY. *Let E be a Banach space with. l.u.st. Then the following assertions are equivalent* :

(i) *E contains no isomorphic copy of $\mathbf{l}_1$*

(ii) *E' has the Radon-Nikodym property i.e. every absolutely summing operator from a space $C(S)$ into E' is nuclear.*

Proof. (i) $\Rightarrow$ (ii) If $T \in \mathscr{L}(C(S), E')$ is absolutely summing then T admits a factorization $C(S) \xrightarrow{i} L_1(\mu) \xrightarrow{u} E'$ where μ is a positive Radon measure on S and i denotes the canonical inclusion. See Proposition 2.3.4 in [24]. Clearly i is weakly compact and $T'' = (i' \circ U'|E)'$. By Theorem 1 in [7] it follows that $i' : L_\infty(\mu) \to C(S)'$ maps weak Cauchy sequences into converging sequences. Because E contains no isomorphic copy of $\mathbf{l}_1$, each bounded sequence in E has a weak Cauchy subsequence (see [27]) and thus $i' \circ U'\ |E$ and T are compact operators. Consequently we have only to prove that each separable subspace of E' has the Radon-Nikodym property.

It is easy to show that each separable subspace of E' embeds into a space F' for F a suitable separable sub-space of E. It was noted in [9], page 134, that each separable dual space has the Radon-Nikodym property and thus in order to prove (ii) it suffices to prove that each separable subspace of E has a separable dual. On the other hand if Z has *l.u.st.* and Y is a separable subspace of Z then there exists a separable Banach space X with *l.u.st.* such that $Y \subset X \subset Z$. Use Definition 3.2 (ii) above. Then we can assume that E itself is separable and our result follows from Theorem 3.3.

(ii) $\Rightarrow$ (i). Since the canonical inclusion $C[0,1] \to L_1[0,1]$ is not nuclear but absolutely summing, E$'$ contains no isomorphic copy of $L_1[0,1]$, which in turn implies (see [23]) that E contains no isomorphic copy of $\mathbf{l}_1$, q.e.d.

3.5. Remark. A Banach space E with local unconditional structure is reflexive if (and only if) $\mathbf{l}_1$ does not isomorphicaly embed in E and E'.

Consequently, a separable Banach space E with local unconditional structure is reflexive if E' contains no isomorph of $\mathbf{l}_1$, if E'' is separable.

Proof. Since E contains no isomorph of $\mathbf{l}_1$ then E' contains no isomorph of $\mathbf{c}_0$. See [1] for details. Then Theorem 1 in [30] and Theorem 13 in [29] together imply that E' is weakly sequentially complete. By [27] any bounded subset of E' is weakly sequentially precompact, and our result follows.

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